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SOME EMBEDDINGS INTO TOTAL MORREY-GULIYEV SPACES ON CARLESON CURVES

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Abstract. In this paper, we study some embeddings into the Morrey spaces $L_{p,\lambda}(\Gamma)$, modified Morrey spaces $\widetilde{L}_{p,\lambda}(\Gamma)$, and total Morrey-Guliyev spaces $L_{p,\lambda,\mu}(\Gamma)$ on Carleson curves Γ.

Keywords: Carleson curves, Morrey spaces, modified Morrey spaces, total Morrey-Guliyev spaces, embeddings.

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1. Introduction

Morrey spaces, introduced by Morrey [26], play an important role in the regularity theory of PDE, including heat equations and Navier-Stokes equations. In harmonic analysis, Morrey spaces are crucial for an alyzing the behavior of integral operators and providing conditions for the global existence of solutions to nonlinear PDEs, such as the Schrödinger equation. The total Morrey-Guliyev spaces $L_{p,\lambda,\mu}(R^n)$, introduced by Guliyev [11], extend the Morrey space $L_{p,\lambda}(R^n)$ by including the second parameter μ, which can be seen as the intermediate spaces between Lebesgue spaces and Morrey spaces. The norm in these spaces is defined by a combination of the norms of $L_{p,\lambda}(R^n)$ and $L_{p,\mu}(R^n)$, which allows a wider range of behavior. Let $0 < p < \infty$, $\lambda \in R$, $\mu \in R$, $[t]_1 = \min\{1,t\}$, $t > 0$. The total Morrey-Guliyev spaces $L_{p,\lambda,\mu}(R^n)$ are the set of all locally integrable functions f with the finite (quasi-)norm

$$\|f\|_{L_{p,\lambda,\mu}(R^n)} = \sup_{x\in R^n, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{L_p(B(x,t))},$$

where $B(x,t)$ denotes the ball centered at x with radius $t > 0$. Here the norm in the case $\mu \le \lambda$ is equal to the maximum of the norms of $L_{p,\lambda}(R^n)$ and $L_{p,\mu}(R^n)$, see also [1, 2, 8, 22, 24, 25, 27, 28, 29, 30, 32, 33, 34]. Total Morrey-Guliyev spaces can be viewed as generalizations of both classical and modified Morrey spaces. In particular, the case where $\lambda = \mu$ corresponds to classical Morrey space, and the case where $\mu = 0$ corresponds to modified Morrey space $\widetilde{L}_{p,\lambda}\left(R^n\right)$, see [3, 4, 7, 10, 12, 13, 14, 15, 16, 20, 21, 28].

Let $\Gamma = \{t \in C : t = t(s), 0 \le s \le l \le \infty\}$ be a rectifiable Jordan curve in the complex plane with arc-length measure $\nu(t) = s$, here $l = \nu\Gamma =$ lengths of Γ.

We denote

$$\Gamma(t,r) = \Gamma \cap B(t,r),\ t \in \Gamma,\ r > 0,$$

where $B(t,r) = \{z \in C : |z - t| < r\}$.

A rectifiable Jordan curve Γ is called a Carleson curve (regular curve) if the condition

$$\nu\,\Gamma(t,r) \le c_0\, r$$

holds for all $t \in \Gamma$ and $r > 0$, where the constant $c_0 > 0$ does not depend on t and r.

It is well known that maximal operator play an important role in harmonic analysis (see [5, 6, 17, 18, 19, 31]). In this paper, in the framework of this analysis in the setting of Carleson curve, we study some embeddings into the Morrey spaces $L_{p,\lambda}(\Gamma)$, modified Morrey spaces $\widetilde{L}_{p,\lambda}(\Gamma)$, and total Morrey-Guliyev spaces $L_{p,\lambda,\mu}(\Gamma)$ on Carleson curves Γ.

The paper is organized as follows. In Section 2, we present some definitions and auxiliary results. In Section 3, we give some embeddings into the total Morrey-Guliyev spaces on Carleson curves.

By A ≲ B we mean that A ≤ CB with some positive constant C independent of appropriate quantities. If A ≲ B and B ≲ A, we write A ≈ B and say that A and B are equivalent.

2. Preliminaries in the setting on Carleson curves and some auxiliary lemmas

Definition 1. *Let* $0 < p < \infty$, $\lambda \in R$, $\mu \in R$, $[r]_1 = \min\{1, r\}, r > 0$. *We denote by* $L_{p,\lambda}(\Gamma)$ *the Morrey space [23], by* $\widetilde{L}_{p,\lambda}(\Gamma)$ *the modified Morrey space [9], and by* $L_{p,\lambda,\mu}(\Gamma)$ *the total Morrey-Guliyev space [11], associated with the Carleson curves the set of all classes of locally integrable functions* f *with the finite norms*

$$\|f\|_{L_{p,\lambda}(\Gamma)} = \sup_{t\in\Gamma, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(\Gamma(t,r))},$$

$$\|f\|_{\widetilde{L}_{p,\lambda}(\Gamma)} = \sup_{t\in\Gamma, r>0} [r]_1^{-\frac{\lambda}{p}} \|f\|_{L_p(\Gamma(t,r))}$$

$$\|f\|_{L_{p,\lambda,\mu}(\Gamma)} = \sup_{t\in\Gamma, r>0} [r]_1^{-\frac{\lambda}{p}} [1/r]_1^{\frac{\mu}{p}} \|f\|_{L_p(\Gamma(t,r))}$$

respectively.

Definition 2. *Let* $0 < p < \infty$, $\lambda \in R$ *and* $\mu \in R$. *We define the weak Morrey spaces* $L_{p,\lambda}(\Gamma)$ *[23] (≡ weak* D_ν *-Morrey space), the weak modified Morrey space* $\widetilde{L}_{p,\lambda}(\Gamma)$ *[9], and the weak total Morrey-Guliyev space* $L_{p,\lambda,\mu}(\Gamma)$ *[11], associated with the Carleson curves the set of all classes of locally integrable functions* f *with the finite norms*

$$\|f\|_{WL_{p,\lambda}(\Gamma)} = \sup_{t\in\Gamma, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(\Gamma(t,r))},$$

$$\|f\|_{W\widetilde{L}_{p,\lambda}(\Gamma)} = \sup_{t\in\Gamma, r>0} [r]_1^{-\frac{\lambda}{p}} \|f\|_{WL_p(\Gamma(t,r))}$$

$$\|f\|_{WL_{p,\lambda,\mu}(\Gamma)} = \sup_{t\in\Gamma, r>0} [r]_1^{-\frac{\lambda}{p}} [1/r]_1^{\frac{\mu}{p}} \|f\|_{WL_p(\Gamma(t,r))}$$

respectively.

Note that

$$L_{p,0,0}(\Gamma) = \widetilde{L}_{p,0}(\Gamma) = L_{p,0}(\Gamma) = L_p(\Gamma),$$

$$WL_{p,0,0}(\Gamma) = W\widetilde{L}_{p,0}(\Gamma) = WL_{p,0}(\Gamma) = WL_p(\Gamma),$$

$$L_{p,\lambda,\lambda}(\Gamma) = L_{p,\lambda}(\Gamma), \quad L_{p,\lambda,0}(\Gamma) = \widetilde{L}_{p,\lambda}(\Gamma),$$

$$\|f\|_{WL_{p,\lambda,\mu}(\Gamma)} \le \|f\|_{L_{p,\lambda,\mu}(\Gamma)} \quad \text{and therefore} \quad L_{p,\lambda,\mu}(\Gamma) \subset WL_{p,\lambda,\mu}(\Gamma)$$

and

$$L_{p,\lambda,\mu}(\Gamma) \subset_{\succ} L_{p,\lambda}(\Gamma), \ \mu \le \lambda \ \text{ and } \ \|f\|_{L_{p,\lambda}(\Gamma)} \le \|f\|_{L_{p,\lambda,\mu}(\Gamma)} \ , \tag{1}$$

$$L_{p,\lambda,\mu}(\Gamma) \subset_{\succ} L_{p,\mu}(\Gamma), \ \mu \le \lambda \ \text{ and } \ \|f\|_{L_{p,\mu}(\Gamma)} \le \|f\|_{L_{p,\lambda,\mu}(\Gamma)} \ , \tag{2}$$

$$\widetilde{L}_{p,\lambda}(\Gamma) \subset_{\succ} L_p(\Gamma) \ \text{ and } \ \|f\|_{L_p(\Gamma)} \le \|f\|_{\widetilde{L}_{p,\lambda}(\Gamma)} \ ,$$

and if $\lambda < 0$ or $\lambda > 1$, then $L_{p,\lambda}(\Gamma) = \widetilde{L}_{p,\lambda}(\Gamma) = WL_{p,\lambda}(\Gamma) = W\widetilde{L}_{p,\lambda}(\Gamma) = \Theta$. Here $\Theta \equiv \Theta(\Gamma)$ is the set of all functions on Γ that are equivalent to 0 on Γ.

To prove our theorems we need the following lemmas.

Lemma 1. *Let* Γ *be a Carleson curve. If* $0 < p < \infty$, $0 \le \lambda \le 1$ *and* $0 \le \mu \le 1$, *then for* $1 \le p < \infty$, $L_{p,\lambda,\mu}(\Gamma)$ *is a Banach space and for* $0 < p < 1$, $L_{p,\lambda,\mu}(\Gamma)$ *is a quasi-Banach space.*

Proof. If $1 \le p < \infty$, then $L_{p,\lambda,\mu}(\Gamma)$ is a normed space. If $0 < p < 1$, recall that for $f, g \in L_p(\Gamma(t,r))$

$$\|f+g\|_{L_p(\Gamma(t,r))} \le 2^{\frac{1}{p}-1}\left(\|f\|_{L_p(\Gamma(t,r))} + \|g\|_{L_p(\Gamma(t,r))}\right).$$

Hence for $0 < p < 1$, $f, g \in L_{p,\lambda,\mu}(\Gamma)$

$$\|f+g\|_{L_{p,\lambda,\mu}(\Gamma)} \le 2^{\frac{1}{p}-1}\left(\|f\|_{L_{p,\lambda,\mu}(\Gamma)} + \|g\|_{L_{p,\lambda,\mu}(\Gamma)}\right) \tag{3}$$

Thus for $0 < p < 1$, $L_{p,\lambda,\mu}(\Gamma)$ is a quasi-normed space (and not a normed space).

Let $f_k \in L_{p,\lambda,\mu}(\Gamma)$, $k \in N$, and

$$\lim_{k,m\to\infty} \|f_k - f_m\|_{L_{p,\lambda,\mu}(\Gamma)} = 0.$$

Then for any $\varepsilon > 0$ there exists $k_0 \in N$ such that for all $k, m \in N$, $k, m \ge k_0$ and for all $t \in \Gamma$, $r > 0$

$$[r]_1^{-\frac{\lambda}{p}}[1/r]_1^{\frac{\mu}{p}}\|f_k - f_m\|_{L_p(\Gamma(t,r))} \le \|f_k - f_m\|_{L_{p,\lambda,\mu}(\Gamma)} < \varepsilon. \tag{4}$$

Hence for all $t \in \Gamma$ and $r > 0$

$$\lim_{k,m\to\infty} \|f_k - f_m\|_{L_p(\Gamma(t,r))} = 0.$$

Due to the completeness of $L_{loc}^p(\Gamma)$ there exists a function $f \in L_{loc}^p(\Gamma)$ such that for all $t \in \Gamma$ and $r > 0$

$$\lim_{k\to\infty} \|f_k - f\|_{L_p(\Gamma(t,r))} = 0.$$

Passing in (4) to the limit as $m \to \infty$ we get that for all $t \in \Gamma$ and $r > 0$

$$[r]_1^{-\frac{\lambda}{p}}[1/r]_1^{\frac{\mu}{p}}\|f_k - f\|_{L_p(\Gamma(t,r))} \le \varepsilon,$$

hence

$$\|f_k - f\|_{L_{p,\lambda,\mu}(\Gamma)} \le \varepsilon,$$

which means that $\lim_{k\to\infty} \|f_k - f\|_{L_{p,\lambda,\mu}(\Gamma)} = 0$. So the space $L_{p,\lambda,\mu}(\Gamma)$ is complete.

Lemma 2. *Let* Γ *be a Carleson curve,* $1 \le p < \infty$, $0 \le \lambda \le 1$ *and* $0 \le \mu \le 1$. *For* $r > 0$ *and* $t = t(s) \in \Gamma$ *define* $rt = t(rs)$, $f_r(t) =: f(rt)$, $[r]_{1,+} = \max\{1, r\}$. *Then*

$$\|f_r\|_{L_{p,\lambda}(\Gamma)} = r^{-\frac{1-\lambda}{p}} \|f\|_{L_{p,\lambda}(\Gamma)},$$

$$\|f_r\|_{\tilde{L}_{p,\lambda}(\Gamma)} = r^{-\frac{1}{p}} [r]_{1,+}^{\frac{\lambda}{p}} \|f\|_{\tilde{L}_{p,\lambda}(\Gamma)},$$

$$\|f_r\|_{L_{p,\lambda,\mu}(\Gamma)} = r^{-\frac{1}{p}} [r]_{1,+}^{-\frac{\lambda}{p}} [1/r]_{1,+}^{-\frac{\mu}{p}} \|f\|_{L_{p,\lambda,\mu}(\Gamma)}.$$

Proof. Note that for $r > 0$ the following equality is true $\sup\limits_{s>0} \frac{[rs]_1}{[s]_1} = [r]_{1,+}$.

Indeed, let $0 < r \le 1$. Then,

$$\sup_{s>0} \frac{[rs]_1}{[s]_1} = \max\left\{ \sup_{0<s\le\frac{1}{r}} \frac{[rs]_1}{[s]_1}, \sup_{s>\frac{1}{r}} \frac{[rs]_1}{[s]_1} \right\} = \max\left\{ \sup_{0<s\le\frac{1}{r}} \frac{[rs]_1}{[s]_1}, 1 \right\}$$

$$= \max\left\{ \max\left\{ \sup_{0<s\le 1} \frac{[rs]_1}{[s]_1}, \sup_{1<s<\frac{1}{r}} \frac{[rs]_1}{[s]_1} \right\}, 1 \right\} = \max\{\max\{r,1\},1\} = 1.$$

Let now $r > 1$. Then

$$\sup_{s>0} \frac{[rs]_1}{[s]_1} = \max\left\{ \sup_{0<s\le\frac{1}{r}} \frac{[rs]_1}{[s]_1}, \sup_{s>\frac{1}{r}} \frac{[rs]_1}{[s]_1} \right\} = \max\left\{ r, \sup_{s>\frac{1}{r}} \frac{[rs]_1}{[s]_1} \right\}$$

$$= \max\left\{ r, \max\left\{ \sup_{\frac{1}{r}<s\le 1} \frac{[rs]_1}{[s]_1}, \sup_{s>1} \frac{[rs]_1}{[s]_1} \right\} \right\} = \max\left\{ r, \max\left\{ \sup_{\frac{1}{r}<s\le 1} \frac{1}{s}, 1 \right\} \right\}$$

$$= \max\{r, \max\{r,1\}\} = r .$$

Next

$$\|f_r\|_{L_{p,\lambda}(\Gamma)} = r^{-\frac{1}{p}} \sup_{t\in\Gamma, s>0} \left(s^{-\lambda} \int_{\Gamma(rt,rs)} |f(\tau)|^p \, d\nu(\tau) \right)^{1/p} = r^{-\frac{1-\lambda}{p}} \|f\|_{p,\lambda(\Gamma)},$$

$$\|f_r\|_{\tilde{L}_{p,\lambda}(\Gamma)} = r^{-\frac{1}{p}} \sup_{t\in\Gamma, s>0} \left([s]_1^{-\lambda} \int_{\Gamma(rt,rs)} |f(\tau)|^p \, d\nu(\tau) \right)^{1/p}$$

$$= r^{-\frac{1}{p}} \sup_{s>0} \left(\frac{[rs]_1}{[s]_1} \right)^{\lambda/p} \sup_{t\in\Gamma, s>0} \left([rs]_1^{-\lambda} \int_{\Gamma(rt,rs)} |f(\tau)|^p \, d\nu(\tau) \right)^{1/p} = r^{-\frac{1}{p}} [r]_{1,+}^{\frac{\lambda}{p}} \|f\|_{\tilde{L}_{p,\lambda}(\Gamma)}$$

and

$$\|f_r\|_{L_{p,\lambda,\mu}(\Gamma)} = r^{-\frac{1}{p}} \sup_{t\in\Gamma, s>0} \left([s]_1^{-\lambda} [1/s]_1^{-\mu} \int_{\Gamma(rt,rs)} |f(\tau)|^p \, d\nu(\tau) \right)^{1/p}$$

$$= r^{-\frac{1}{p}} \sup_{s>0}\left(\frac{[rs]_1}{[s]_1}\right)^{\lambda/p} \sup_{s>0}\left(\frac{[1/(rs)]_1}{[1/s]_1}\right)^{-\mu/p} \sup_{t\in\Gamma,\, s>0}\left([rs]_1^{-\lambda}\int_{\Gamma(rt,rs)}|f(\tau)|^p d\nu(\tau)\right)^{1/p}$$

$$= r^{-\frac{1}{p}}[r]_{1,+}^{\frac{\lambda}{p}}[1/r]_{1,+}^{-\frac{\mu}{p}}\|f\|_{L_{p,\lambda,\mu}(\Gamma)}.$$

Lemma 3. *Let Γ be a Carleson curve, $1\le p<\infty$, $0\le\mu\le\lambda\le1$. Then*

$$L_{p,\lambda,\mu}(\Gamma)=L_{p,\lambda}(\Gamma)\cap L_{p,\mu}(\Gamma)$$

and

$$\|f\|_{L_{p,\lambda,\mu}(\Gamma)}=\max\left\{\|f\|_{L_{p,\lambda}(\Gamma)},\|f\|_{L_{p,\mu}(\Gamma)}\right\}.$$

Proof. Let $f\in L_{p,\lambda,\mu}(\Gamma)$. Then from (1) and (2) we have that $f\in L_{p,\lambda}(\Gamma)\cap L_{p,\mu}(\Gamma)$ and $\max\left\{\|f\|_{L_{p,\lambda}(\Gamma)},\|f\|_{L_{p,\mu}(\Gamma)}\right\}\le\|f\|_{L_{p,\lambda,\mu}(\Gamma)}$.

Let now $f\in L_{p,\lambda}(\Gamma)\cap L_{p,\mu}(\Gamma)$. Then

$$\|f\|_{L_{p,\lambda,\mu}(\Gamma)}=\sup_{t\in\Gamma,\, r>0}\left([r]_1^{-\lambda}[1/r]_1^{\lambda}\int_{\Gamma(t,r)}|f(\tau)|^p d\nu(\tau)\right)^{1/p}$$

$$=\max\left\{\sup_{t\in\Gamma,\,0<r\le1}\left(r^{-\lambda}\int_{\Gamma(t,r)}|f(\tau)|^p d\nu(\tau)\right)^{1/p},\ \sup_{t\in\Gamma,\, r>1}\left(r^{-\mu}\int_{\Gamma(t,r)}|f(\tau)|^p d\nu(\tau)\right)^{1/p}\right\}$$

$$\le\max\left\{\|f\|_{L_{p,\lambda}(\Gamma)},\|f\|_{L_{p,\mu}(\Gamma)}\right\}.$$

Therefore, $f\in L_{p,\lambda,\mu}(\Gamma)$ and the embedding $L_{p,\lambda}(\Gamma)\cap L_{p,\mu}(\Gamma)\subset_{\succ} L_{p,\lambda,\mu}(\Gamma)$ is valid.

Thus $L_{p,\lambda,\mu}(\Gamma)=L_{p,\lambda}(\Gamma)\cap L_{p,\mu}(\Gamma)$ and $\max\left\{\|f\|_{L_{p,\lambda}(\Gamma)},\|f\|_{L_{p,\mu}(\Gamma)}\right\}=\|f\|_{L_{p,\lambda,\mu}(\Gamma)}$.

Corollary 1. *Let Γ be a Carleson curve, $1\le p<\infty$, $0\le\lambda\le1$. Then*

$$\widetilde{L}_{p,\lambda}(\Gamma)=L_{p,\lambda}(\Gamma)\cap L_p(\Gamma)$$

and

$$\|f\|_{\widetilde{L}_{p,\lambda}(\Gamma)}=\max\left\{\|f\|_{L_{p,\lambda}(\Gamma)},\|f\|_{L_p(\Gamma)}\right\}.$$

Remark 1. *If $1\le p<\infty$, and $\mu<0$ or $\lambda>1$, then*

$$L_{p,\lambda,\mu}(\Gamma)=WL_{p,\lambda,\mu}(\Gamma)=\Theta(\Gamma).$$

Example 1. If $\alpha>-\frac{1}{p}$, $0<p<\infty$ and $0\le\lambda\le\mu\le1$, then

$$|\cdot|^{\alpha}\in L_{p,\lambda,\mu}(\Gamma)\Leftrightarrow-\frac{1-\lambda}{p}\le\alpha\le-\frac{1-\mu}{p},$$

$$|\cdot|^{\alpha}\chi_{\Gamma(0,1)}\in L_{p,\lambda,\mu}(\Gamma)\Leftrightarrow\alpha\ge-\frac{1-\lambda}{p},$$

and

$$|\cdot|^{\alpha}\chi_{{}^{c}\Gamma(0,1)}\in L_{p,\lambda,\mu}(\Gamma)\Leftrightarrow\alpha\le-\frac{1-\mu}{p}.$$

Indeed,

$$\left\||\cdot|^{\alpha}\right\|_{L_{p,\lambda,\mu}(\Gamma)}=\sup_{t\in\Gamma,\,r>0}[r]_1^{-\frac{\lambda}{p}}[1/r]_1^{\frac{\mu}{p}}\left\||\tau|^{\alpha}\right\|_{L_p(\Gamma(t,r))}$$

$$=\sup_{r>0}[r]_1^{-\frac{\lambda}{p}}[1/r]_1^{\frac{\mu}{p}}\left\||\tau|^{\alpha}\right\|_{L_p(\Gamma(0,r))}$$

$$=\sup_{r>0}[r]_1^{-\frac{\lambda}{p}}[1/r]_1^{\frac{\mu}{p}}\left(\int_{\Gamma(0,r)}|\tau|^{\alpha p}\nu(\tau)\right)^{\frac{1}{p}}$$

$$=\left(\frac{1}{\alpha p}\right)^{\frac{1}{p}}\sup_{r>0}[r]_1^{-\frac{\lambda}{p}}[1/r]_1^{\frac{\mu}{p}}r^{\alpha+\frac{1}{p}}$$

$$=\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\max\left\{\sup_{0<r\le1}r^{\alpha+\frac{1-\lambda}{p}},\sup_{r>1}r^{\alpha+\frac{1-\mu}{p}}\right\}=\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}.$$

The following statement can be proved analogously.

Lemma 4. *Let* Γ *be a Carleson curve,* $1\le p<\infty$, $0\le\mu\le\lambda\le1$. *Then*

$$W\widetilde{L}_{p,\lambda}(\Gamma)=WL_{p,\lambda}(\Gamma)\cap WL_{p,\mu}(\Gamma)$$

and

$$\|f\|_{W\widetilde{L}_{p,\lambda}(\Gamma)}=\max\left\{\|f\|_{WL_{p,\lambda}(\Gamma)},\|f\|_{WL_{p,\mu}(\Gamma)}\right\}.$$

3. Some embeddings into total Morrey-Guliyev spaces on Carleson curves

In this section, we study some embeddings into the Morrey spaces $L_{p,\lambda}(\Gamma)$, modified Morrey spaces $\widetilde{L}_{p,\lambda}(\Gamma)$, and total Morrey-Guliyev spaces $L_{p,\lambda,\mu}(\Gamma)$ on Carleson curves Γ.

Lemma 5. *Let* Γ *be a Carleson curve,* $1\le p<\infty$, $0\le\lambda_2\le\lambda_2\le1$ *and* $0\le\mu_2\le\mu_2\le1$, *then* $L_{p,\lambda_1,\mu_1}(\Gamma)\subset_{\succ}L_{p,\lambda_2,\mu_2}(\Gamma)$

and

$$\|f\|_{L_{p,\lambda_2,\mu_2}(\Gamma)}\le L_{p,\lambda_1,\mu_1}(\Gamma).$$

Proof. Let $f\in L_{p,\lambda_1,\mu_1}(\Gamma), 1\le p<\infty, 0\le\lambda_2\le\lambda_1\le1,\ 0\le\mu_1\le\mu_2\le1$. Then

$$\|f\|_{L_{p,\lambda_2,\mu_2}(\Gamma)}=\max\left\{\sup_{t\in\Gamma,0<r\le 1} r^{-\frac{\lambda_1-\lambda_2}{p}} r^{-\frac{\lambda_1}{p}}\|f\|_{L_p(\Gamma(t,r))},\ \sup_{t\in\Gamma,r\ge 1} r^{-\frac{\mu_1-\mu_2}{p}} r^{-\frac{\mu_1}{p}}\|f\|_{L_p(\Gamma(t,r))}\right\}\le\|f\|_{L_{p,\lambda_1,\mu_1}(\Gamma)}.$$

Lemma 6. *Let Γ be a Carleson curve, $1\le p<\infty$, $0\le\lambda\le 1$ and $0\le\mu\le 1$, then*

$$L_{p,1,\mu}(\Gamma)\subset_{\succ} L_\infty(\Gamma)\subset_{\succ} L_{p,\lambda,1}(\Gamma)$$

and

$$\|f\|_{L_{p,\lambda,1}(\Gamma)}\le c_0^{\frac{1}{p}}\|f\|_{L_\infty(\Gamma)}\le\|f\|_{L_{p,1,\mu}(\Gamma)}.$$

Proof. Let $f\in L_\infty(\Gamma)$. Then for all $t\in\Gamma$ and $0<r\le 1$

$$r^{-\frac{\lambda}{p}}\|f\|_{L_p(\Gamma(t,r))}\le c_0^{\frac{1}{p}}\|f\|_{L_\infty(\Gamma)},\ \ 0\le\lambda\le 1$$

and for all $t\in\Gamma$ and $r\ge 1$

$$r^{-\frac{1}{p}}\|f\|_{L_p(\Gamma(t,r))}\le c_0^{\frac{1}{p}}\|f\|_{L_\infty(\Gamma)}.$$

Thus $f\in L_{p,\lambda,1}(\Gamma)$ and

$$\|f\|_{L_{p,\lambda,1}(\Gamma)}\le c_0^{\frac{1}{p}}\|f\|_{L_\infty(\Gamma)}.$$

Let $f\in L_{p,1,\mu}(\Gamma)$. By the Lebesgue's differentiation theorem we have

$$\lim_{t\to 0}|\Gamma(t,r)|^{-\frac{1}{p}}\|f\|_{L_p(\Gamma(t,r))}=|f(t)|\ \text{ for a.e. } t\in\Gamma.$$

Then for a.e. $t\in\Gamma$.

$$|f(t)|=|\Gamma(t,r)|^{-\frac{1}{p}}\|f\|_{L_p(\Gamma(t,r))}\le c_0^{-\frac{1}{p}}\sup_{t\in\Gamma,0<r\le 1} r^{-\frac{1}{p}}\|f\|_{L_p(\Gamma(t,r))}\le c_0^{-\frac{1}{p}}\|f\|_{L_{p,1,\mu}(\Gamma)}.$$

Thus $f\in L_\infty(\Gamma)$ and

$$\|f\|_{L_\infty(\Gamma)}\le c_0^{-\frac{1}{p}}\|f\|_{L_{p,1,\mu}(\Gamma)}.$$

Corollary 2. [23] *Let Γ be a Carleson curve and $1\le p<\infty$. Then*

$$L_{p,1}(\Gamma)=L_\infty(\Gamma)$$

and

$$\|f\|_{L_\infty(\Gamma)}\le\|f\|_{L_{p,1}(\Gamma)}\le c_0^{1/p}\|f\|_{L_\infty(\Gamma)}.$$

Lemma 7. *Let Γ be a Carleson curve, $0\le\lambda<1$, $0\le\mu<1$, $0\le\alpha<1-\lambda$ and $0\le\beta<1-\mu$, then for $\frac{1-\lambda}{\alpha}\le p\le\frac{1-\mu}{\beta}$*

$$L_{p,\lambda,\mu}(\Gamma) \subset\!\!\succ L_{1,1-\alpha,1-\beta}(\Gamma)$$

and for $f \in L_{p,\lambda,\mu}(\Gamma)$ *the inequality*

$$\|f\|_{L_{1,1-\alpha,1-\beta}(\Gamma)} \le c_0^{\frac{1}{p'}} \|f\|_{L_{p,\lambda,\mu}(\Gamma)}$$

holds.

Proof. Let $0<\alpha<1$, $0\le\lambda<1$, $f \in L_{p,\lambda,\mu}(\Gamma)$ and $\dfrac{1-\lambda}{\alpha} \le p \le \dfrac{1-\mu}{\beta}$. By the Hölder's inequality we have

$$\|f\|_{L_{1,1-\alpha,1-\beta}(\Gamma)} = \sup_{t\in\Gamma, t>0} [r]_1^{\alpha-1} [1/r]_1^{1-\beta} \|f\|_{L_1(\Gamma(t,r))}$$

$$\sup_{t\in\Gamma, r>0} [r]_1^{\alpha-1} [1/r]_1^{1-\beta} \|f\|_{L_p(\Gamma(t,r))} \|1\|_{L_{p'}(\Gamma(t,r))}$$

$$\le c_0^{\frac{1}{p'}} \sup_{t\in\Gamma, r>0} \left([r]_1 r^{-1}\right)^{-\frac{1}{p'}} [r]_1^{\alpha-\frac{1-\lambda}{p}} [1/r]_1^{1-\beta-\frac{\mu}{p}} [r]_1^{-\frac{\lambda}{p}} [1/r]_1^{\frac{\mu}{p}} \|f\|_{L_p(\Gamma(t,r))}$$

$$\le c_0^{\frac{1}{p'}} \|f\|_{L_{p,\lambda,\mu}(\Gamma)} \sup_{r>0} \left([r]_1 r^{-1}\right)^{\frac{1-\mu}{p}-\beta} [r]_1^{\alpha-\frac{1-\lambda}{p}}.$$

Note that

$$\sup_{r>0} \left([r]_1 r^{-1}\right)^{\frac{1-\mu}{p}-\beta} [r]_1^{\alpha-\frac{1-\lambda}{p}} = \max\left\{ \sup_{0<r\le 1} r^{\alpha-\frac{1-\lambda}{p}}, \sup_{r>1} r^{\beta-\frac{1-\mu}{p}} \right\} < \infty$$

$$\Leftrightarrow \frac{1-\lambda}{\alpha} \le p \le \frac{1-\mu}{\beta}.$$

Thus $f \in L_{1,1-\alpha,1-\beta}(\Gamma)$ and

$$\|f\|_{L_{1,1-\alpha,1-\beta}(\Gamma)} \le c_0^{-\frac{1}{p'}} \|f\|_{L_{p,1,\mu}(\Gamma)}.$$

From Lemma 7 we obtain the following results.

Corollary 3. *Let* Γ *be a Carleson curve,* $0\le\mu\le\lambda<1$, $0\le\alpha<1-\lambda$, *then for* $\dfrac{1-\lambda}{\alpha} \le p \le \dfrac{1-\mu}{\beta}$.

$$L_{p,\lambda,\mu}(\Gamma) \subset\!\!\succ L_{1,1-\alpha}(\Gamma)$$

and for $f \in L_{p,\lambda,\mu}(\Gamma)$ *the inequality*

$$\|f\|_{L_{1,1-\alpha}(\Gamma)} \le c_0^{-\frac{1}{p'}} \|f\|_{L_{p,\lambda,\mu}(\Gamma)}$$

holds.

Corollary 4. [23] *Let* Γ *be a Carleson curve,* $1 \le p < \infty$ *and* $0 \le \lambda < 1$. *If* $p = \frac{1-\lambda}{\alpha}$, *then*

$$L_{p,\lambda}(\Gamma) \subset L_{1,1-\alpha}(\Gamma) \qquad \text{and} \qquad \|f\|_{L_{1,1-\alpha}(\Gamma)} \le c_0^{1/p'} \|f\|_{L_{p,\lambda}(\Gamma)} ,$$

where $1/p + 1/p' = 1$.

Corollary 5. *Let* Γ *be a Carleson curve,* $0 < \alpha < 1$, $0 \le \lambda \le 1-\alpha$. *Then for* $\frac{1-\lambda}{\alpha} \le p \le \frac{1}{\alpha}$

$$\widetilde{L}_{p,\lambda}(\Gamma) \subset L_{1,1-\alpha}(\Gamma) \qquad \text{and} \qquad \|f\|_{L_{1,1-\alpha}(\Gamma)} \le c_0^{1/p'} \|f\|_{\widetilde{L}_{p,\lambda}(\Gamma)} .$$

4. Conclusion

We find some embeddings into the Morrey spaces $L_{p,\lambda}(\Gamma)$, modified Morrey spaces $\widetilde{L}_{p,\lambda}(\Gamma)$, and total Morrey-Guliyev spaces $L_{p,\lambda,\mu}(\Gamma)$ on Carleson curves Γ.

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